Philosophy 324A
Philosophy of Logic
2016

Note Two

Some Examples of Modal Axiomatics

Please memorize the axioms and rules for S1, S2, S4 and S5.

1**.** *The Lewis Systems, 1912-1932***.** C.I. Lewis and C.H. Langford, *Symbolic Logic*, New York: Century 1932; New York: Dover 1959. See also Lewis, *A Survey of Symbolic Logic,* Berkeley and Los Angeles, University of California Press, 1918.

Rules

Substitution. If A′ is exactly like A except for containing a wff C at some places where A contains B then ⊢ (B ≡ C) ⊃ (A′ ≡ A). In words, if A′ and A are the same wff except for different but strictly equivalent parts, then A′ is strictly equivalent to A.

Adjunction. If ⊢A and ⊢B then ⊢(A ∧ B).

Inference. If ⊢A and ⊢(A ⥽ B) then ⊢ B.

Definitions: □A iff ~◊ ~A; ◊A iff ~□ ~A; A ⥽ B iff ~◊(A ∧ ~B)

Axioms for S1

B1. (A ∧ B) ⥽ (B ∧ A)
B2. (A ∧ B) ⥽ A
B3. A ⥽ (A ∧ A)
B 4. ((A ∧ B) ∧ C) ⥽ (A ∧ (B ∧ C))
B6. ((A ⥽ B) ∧ (B ⥽ C)) ⥽ (A ⥽ C)
B7. (A ∧ (A ⥽ B)) ⥽ B

Note: B5 is redundant; it is derivable from B1, 2, 3, 6. See J.C. McKinsey, "A reduction in the number of postulates for C.I. Lewis' system of strict implication", *Bulletin of the American Philosophical Society*, 40, 1934, 425-427.

Axioms for S2

B1-B7 + B8: ◊(A ∧ B) ⥽ ◊A.

Axioms for S3

B1-B7 + A8: (A ⥽ B) ⥽ (~◊B ⥽ ~◊A).

Axioms for S4

B1-B7 + C10: ~◊ ~A ⥽ ~◊ ~~◊ ~A.

Axioms for S5

Axioms of S2 + C11: ◊A ⥽ ~◊~◊A.

Axioms for S6 (M.J. Alban, "Independence of the primitive symbols of Lewis's calculi of propositions", *Journal of Symbolic Logic,* 8, 1943, 25-26).

Axioms of S2 + C13: ◊◊A.

Axioms for S7 Soren Haldèn, ("Results concerning the decision problem of Lewis' calculi S3 and S6", *Journal of Symbolic Logic*, 14, 1950, 230-236).

Axioms for S3 + C13: ◊◊A.

Axioms for S8 (see above)

Axioms for S3 + ~◊ ~◊◊A.

2**.** *The Gödel Systems, 1933* (Kurt Gödel, "Eine Interpretation des intuitionistischen Aussagenkalküls", *Ergebnisse eines mathematischen Kolloquiums* Heft 4, 1933, 39-40)).

Rule

RL: If ⊢A then ⊢□A

Axioms for Gödel's Basic System

A.1: □A ⊃ A.

A.2: □(A ⊃ B) ⊃ (□A ⊃ □B).

Axioms for Gödel's Original System

A.1-A.2 + A.4: □A ⊃ □□A

(Notes: 1. Gödel's Original System is equivalent to S4.

2. Gödel's Basic System when supplemented by the axiom A.5 (◊A ⊃ □◊A) is equivalent to S5.
3. When Gödel's Basic System is supplemented by "Brouwer's" axiom

A.3 ($A \supset \Box\Diamond A$), the result is equivalent to Brouwer's System.

3. *Feys' System, 1937-1938*. (Robert Feys, "Les logiques nouvelles des modalités", *Revue néoscolastique de Philosophie*, 40, 1937, 517-553 and 41, 1938, 217-252).

Rule

Feys' Rule 25.2 is the same as Gödel's RL.

Axioms

Feys' axiom 25.3 is Gödel's A.2
Feys' axiom 23.11 is $A \supset \Diamond A$.

Note: Feys's System is equivalent to Gödel's Basic System.

4. *The von Wright Systems, 1951* (Georg von Wright, *An Essay in Modal Logic,* Amsterdam: North-Holland 1951).

Rules

The rules of a system of classical propositional logic plus:

Extensionality: If $\vdash A \equiv B$ then $\vdash \Diamond A \equiv \Diamond B$.
Tautology: If $\vdash A$ then $\vdash \Box A$.

Axioms for M

The axiom of possibility: $A \supset \Diamond A$
The axiom of distribution: $\Diamond(A \vee B) \equiv (\Diamond A \vee \Diamond B)$.

Axioms for M′

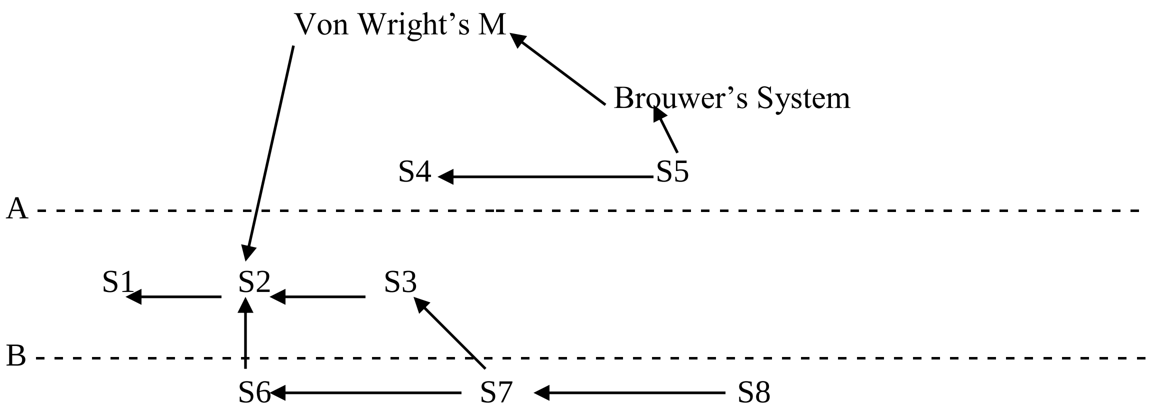
The axioms for M plus:
The first axiom of reduction: $\Diamond\Diamond A \supset \Diamond A$.

Axioms for M″

The axioms for M plus:
The second axiom of reduction: $\Diamond \sim\Diamond A \supset \sim\Diamond A$.

5. *Interrelations Between the Systems*

Gödel's Basic System
Fey's System



1. $\rightarrow$ expresses containment.
2. Systems above line A have rule RL (if $\vdash$A then $\vdash\Box$A)
3. Systems below line A don't have RL.
4. Systems below line B have $\vdash\Diamond\Diamond$A.
5. Systems above line B don't have $\vdash\Diamond\Diamond$A.
6. Systems above line A are incompatible with $\Diamond\Diamond$A.
7. Systems below line B are incompatible with RL.

On the last page of this note is a more recent and comprehensive chart. It is taken with permission and my thanks from Andrew Irvine's "S7", *Journal of Applied Logic*, 11 (2013), 525. For the border key, see the paper.

Further optional reading

I recommend Roberta Ballarin, "Modern origins of modal logic", *Stanford Encyclopedia of Philosophy*, online. Some pre-Lewis developments, can be found in Hugh MacColl, *Symbolic Logic and its Applications*, London: Longmans Green, 1906. MacColl's importance is largely overlooked by the modal mainstream. See here John Woods "MacColl's elusive pluralism", in Amirouche Moktefi and Stephen Read, editors, *Hugh MacColl After One Hundred Years*, pages 205-234, Paris: Editions Kimé, 2011 [= *Philosophia Scientiae*, 15, 2011, 205-234].